

AN INVERSE PROBLEM FOR TRAPPING POINT RESONANCES.

ALEXEI IANTCHENKO

ABSTRACT. We consider semi-classical Schrödinger operator $P(h) = -h^2\Delta + V(x)$ in \mathbb{R}^n such that the analytic potential V has a non-degenerate critical point $x_0 = 0$ with critical value E_0 and we can define resonances in some fixed neighborhood of E_0 when $h > 0$ is small enough. If the eigenvalues of the Hessian are \mathbb{Z} -independent the resonances in h^δ -neighborhood of E_0 ($\delta > 0$) can be calculated explicitly as the eigenvalues of the semi-classical Birkhoff normal form.

Assuming that potential is symmetric with respect to reflections about the coordinate axes we show that the classical Birkhoff normal form determines the Taylor series of the potential at x_0 . As a consequence, the resonances in a h^δ -neighborhood of E_0 determine the first N terms in the Taylor series of V at x_0 .

The proof uses the recent inverse spectral results of V. Guillemin and A. Uribe.

1. INTRODUCTION.

We consider the semi-classical Schrödinger operator

$$(1) \quad P = P(h) = -h^2\Delta + V(x), \quad x \in \mathbb{R}^n,$$

with the symbol $p(x, \xi) = \xi^2 + V(x)$.

If the spectrum of (1) is discrete near some energy E and real-valued potential V is smooth then it is known ([7], [12], [3]) that the spectrum of $P(h)$ in a small fixed neighborhood of E as $h \rightarrow 0$ determines the Birkhoff normal form of the Hamiltonian $p(x, \xi) = \xi^2 + V(x)$. In [1] it was shown that the classical Birkhoff normal form of $p(x, \xi)$ at a non-degenerate minimum x_0 of V determines the Taylor series of the potential provided the eigenvalues of the Hessian are linearly independent over \mathbb{Q} and V satisfies a symmetry condition near x_0 . This result was applied to prove that the low-lying eigenvalues of the semi-classical operator P determine the Taylor series of the potential at x_0 . In this note we study the similar question for the resonances. In [14] it was indicated how the inverse spectral results based on wave invariants translates to inverse results for resonances (see also [13]). We

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consider a special situation as in [9] and [4], when the resonances can be calculated explicitly as the eigenvalues of the semi-classical Birkhoff normal form.

We suppose that general assumptions of Helffer-Sjöstrand in [8] are fulfilled so that we can define resonances in some fixed neighborhood of $E_0 \in \mathbb{R}$ when $h > 0$ is small enough.

We suppose also that V is analytic potential, which extends to a holomorphic function in a set

$$\{x \in \mathbb{C}^n; |\operatorname{Im} x| < \frac{1}{C} \langle \operatorname{Re} x \rangle\}$$

with $V(x) \rightarrow 0$, when $x \rightarrow \infty$ in that set. Here $\langle s \rangle = (1 + |s|^2)^{1/2}$.

We will use notation $\operatorname{neigh}(E, \mathbb{R})$ or $\operatorname{neigh}(E)$ for a real neighborhood of a $E \in \mathbb{R}$.

Following [6] the trapped set $K(E_0)$ is $K(E_0) = \{\rho \in p^{-1}(E_0); \exp tH_p(\rho) \not\rightarrow \infty, t \rightarrow \pm\infty\}$, which is the union of trapped trajectories in $p^{-1}(E_0)$. Here H_p is the Hamilton field of $p(x, \xi)$.

We assume that the union of trapped trajectories in $p^{-1}(E_0)$ is just the point $(0, 0)$:

$$(2) \quad K(E_0) = (0, 0).$$

Then 0 is a unique critical point of V with critical value E_0 . We suppose that 0 is non-degenerate critical point of V with signature $(n - d, d)$:

$$V(0) = E_0, \quad V'(0) = 0, \quad \operatorname{sgn} V''(0) = (n - d, d),$$

so that $V''(0)$ is non-degenerate and

$$(3) \quad V(x) = E_0 + \sum_{j=1}^{n-d} u_j^2 x_j^2 - \sum_{j=n-d+1}^n u_j^2 x_j^2 + \mathcal{O}(|x|^3).$$

Kaidi and Kerdelhue showed in [4] how to adapt the Helffer-Sjöstrand theory and realize $P = -h^2\Delta + V(x)$ as acting in $H(\Lambda)$ -spaces, where $\Lambda \subset \mathbb{C}^{2n}$ is an IR-manifold which coincides with $T^*(\mathbb{R}^{n-d} \oplus e^{i\pi/4}\mathbb{R}^d)$ near $(0, 0)$ and has the property that $\forall \epsilon > 0, \exists \delta > 0$ such that $(x, \xi) \in \Lambda, \operatorname{dist}((x, \xi), (0, 0)) > \epsilon \Rightarrow |p(x, \xi) - E_0| > \delta$.

Then resonances can essentially (modulo an argument using a Grushin reduction) be viewed as an eigenvalue problem for P after the complex scaling $x_j = e^{i\pi/4}\tilde{x}_j, \tilde{x}_j \in \mathbb{R}, n - d + 1 \leq j \leq n$.

We suppose also that the coefficients u_j in (3) satisfy non-resonance condition:

$$(4) \quad \sum_{j=1}^n k_j u_j = 0, \quad k_j \in \mathbb{Z} \Rightarrow k_1 = k_2 = \dots = k_n = 0.$$

Under these assumptions a result of Kaidi and Kerdelhue [4] gives all resonances in a disc $D(E_0, h^\delta)$ of center E_0 and radius h^δ . Here δ can be any fixed constant and $h > 0$ is small enough depending on δ . We cite this result later in Theorem 2.

The consequence of the main result of this note is the following:

Theorem 1. *Assume V is symmetric with respect to reflections about the coordinate axes, i.e. for any choice of signs*

$$(5) \quad V(x_1, \dots, x_n) = V(\pm x_1, \dots, \pm x_n).$$

In addition, assume that

$$(6) \quad V(x) = E_0 + \sum_{j=1}^{n-d} u_j^2 x_j^2 - \sum_{j=n-d+1}^n u_j^2 x_j^2 + \mathcal{O}(|x|^4),$$

where u_1, \dots, u_n are the positive numbers satisfying (4).

Then, given $N > 0$ there exists a $\delta > 0$ such that the resonances in $D(E_0, h^\delta)$ for $0 < h < h_0$, determine the first N terms in the Taylor series of V at zero.

In dimension $n = 1$, $d = 1$, resonances generated by the maximum of the potential (barrier top resonances) are of the form $\simeq V(0) - ih(-V''(x_0)/2)^{1/2}(2k+1) + \dots$, $k = 0, 1, \dots$, $V(0) = E_0$. Yves Colin de Verdière and Victor Guillemin have recently shown in [5] that one can drop the condition that the potential is even. Namely instead of (5) and (6) it is enough to suppose that in the expansion $V(x) = E_0 - ux^2 + \sum_{j=3}^\infty a_j x^j$ the coefficients $u > 0$ and a_3 do not vanish. Then all a_j 's are determined from the coefficients of the quantum Birkhoff normal form once we have chosen the sign of a_3 . The classical Birkhoff normal form alone is not enough to recover the potential.

In dimension $n = 2$, $d = 1$, Sjöstrand (see [11]) showed that the saddle-point resonances are given by the eigenvalues of the Birkhoff normal form in the whole h -independent neighborhood of E_0 . Thus the full Taylor series of V is determined and using the analyticity, the full potential can be recovered from the resonances.

To prove Theorem 1 we use that under non-resonance condition (4) the Schrödinger operator P can be transformed in the semi-classical or quantum Birkhoff normal form (see [10])

$$(7) \quad E_0 + \tilde{P} \equiv U^* P U,$$

where U is analytic unitary Fourier integral operator microlocally defined near $(0, 0)$ and \tilde{P} is pseudodifferential operator with the symbol

$$(8) \quad F \sim \sum_{j=0}^{\infty} h^j F_j(\iota_1, \dots, \iota_{n-d}, j_{n-d+1}, \dots, j_n), \quad \iota_j = \xi_j^2 + x_j^2, \quad j_j = \xi_j^2 - x_j^2,$$

with F_j analytic and principal symbol

$$(9) \quad F_0 = \sum_{j=1}^{n-d} u_j \iota_j + \sum_{j=n-d+1}^n u_j j_j + \mathcal{O}(|(\iota, j)|^2).$$

The equivalence relation \equiv means to infinite order at $(0, 0)$ (see [2]).

The result of [4] shows that, modulo error terms of order $\mathcal{O}(h^\infty)$, the resonances of P in h^δ neighborhood of E_0 are approximated by the eigenvalues of its quantum Birkhoff normal form at $(0, 0)$ after the complex scaling $x_j = e^{i\pi/4}\tilde{x}_j$, $\tilde{x}_j \in \mathbb{R}$, $n - d + 1 \leq j \leq n$, namely

$$\tilde{F} \sim \sum_{j=0}^{\infty} h^j F_j(\iota_1, \dots, \iota_{n-d}, \frac{1}{i}\tilde{\iota}_{n-d+1}, \dots, \frac{1}{i}\tilde{\iota}_n),$$

where F is as in (8) and $\frac{1}{i}\tilde{\iota} = \frac{1}{i}(\tilde{\xi}^2 + \tilde{x}^2) = \xi_j^2 - x_j^2$, $\xi_j = e^{-i\pi/4}\tilde{\xi}_j$, $x_j = e^{i\pi/4}\tilde{x}_j$. We denote $\tilde{F}_j(\iota_1, \dots, \iota_{n-d}, \tilde{\iota}_{n-d+1}, \dots, \tilde{\iota}_n) = F_j(\iota_1, \dots, \iota_{n-d}, \frac{1}{i}\tilde{\iota}_{n-d+1}, \dots, \frac{1}{i}\tilde{\iota}_n)$.

Theorem 2 (Kaidi-Kerdelhue). *The resonances of P in rectangle $]E_0 - \epsilon_0, E_0 + \epsilon_0[-i[0, h^\delta]$ are simple labeled by $k \in \mathbb{N}^n$ and of the form*

$$E_0 + \sum_{j=0}^{\infty} h^j \tilde{F}_j((2k_1 + 1)h, \dots, (2k_n + 1)h)$$

where

$$\tilde{F}_j \in C^\infty(\text{neigh}(0)), \quad \tilde{F}_0(\iota) = \sum_{j=1}^{n-d} u_j \iota_j - \sum_{j=n-d+1}^n i u_j \iota_j + \mathcal{O}(|\iota|^2), \quad \tilde{F}_1(\iota) = V(0) - E_0 = 0.$$

The main result of this note is the following:

Lemma 1. *Assume (4), (5) and (6). Then the classical Birkhoff normal form F_0 determines the Taylor series of V at the origin.*

We show in Section 2 how this lemma follows from [1]. The main idea of the proof is that the complex scaling reduces the principle symbol of P to the form $H(x, \xi) = \sum_{j=1}^n \omega_j(\xi_j^2 + x_j^2) + \mathcal{O}(|x|^3)$ which is similar to the Hamiltonian considered in [1] with the only difference that coefficients ω_j for $n - d + 1 \leq j \leq n$ are complex numbers. We show in Section 2 that the method of Guillemin and Uribe can still be applied.

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2. CLASSICAL BIRKHOFF CANONICAL FORM, PROOF OF LEMMA 1

Conjugating the Hamiltonian $p(x, \xi) = \xi^2 + V(x)$, with V as in (6), by the linear symplectomorphism

$$x_i \mapsto u_i^{1/2} x_i, \quad \xi_i \mapsto u_i^{-1/2} \xi, \quad i = 1, \dots, n,$$

one can assume without loss of generality that

$$p = E_0 + H_1 + V_2 \equiv E_0 + \sum_{j=1}^{n-d} u_j(\xi_j^2 + x_j^2) + \sum_{j=n-d+1}^n u_j(\xi_j^2 - x_j^2) + V_2(x_1^2, \dots, x_n^2),$$

where $V_2(s_1, \dots, s_n) = \mathcal{O}(|s|^2)$. We denote $H = H_1 + V_2$.

Then resonances can essentially (see Introduction) be viewed as an eigenvalue problem for P after the complex scaling

$$(10) \quad x_j = e^{i\pi/4} \tilde{x}_j, \quad \tilde{x}_j \in \mathbb{R}, \quad n-d+1 \leq j \leq n.$$

The principal symbol of the scaled operator becomes $\tilde{p}(\dots, \tilde{x}, \dots, \tilde{\xi}) = E_0 + \tilde{H}_1 + \tilde{V}_2$, where the new $\tilde{H} = \tilde{H}_1 + \tilde{V}_2$ is equal to

$$H(x_1, \dots, x_{n-d}, e^{i\pi/4} \tilde{x}_{n-d+1}, \dots, e^{i\pi/4} \tilde{x}_n, \xi_1, \dots, \xi_{n-d}, e^{-i\pi/4} \tilde{\xi}_{n-d+1}, \dots, e^{-i\pi/4} \tilde{\xi}_n).$$

With $x_j = e^{i\pi/4} \tilde{x}_j$, $\xi_j = e^{-i\pi/4} \tilde{\xi}_j$, for $n-d+1 \leq j \leq n$, we have $\xi_j^2 - x_j^2 = (\tilde{\xi}_j^2 - \tilde{x}_j^2)/i$, and omitting the tildes we get

$$H(x, \xi) = \sum_{j=1}^{n-d} u_j(\xi_j^2 + x_j^2) + \sum_{j=n-d+1}^n \frac{1}{i} u_j(\xi_j^2 + x_j^2) + V_2(x_1^2, \dots, x_n^2)$$

with all x_j, ξ_j real, and which can be identified with the restriction of the old H to the IR-manifold $\Lambda \in \mathbb{C}^{2n}$. Then one can follow Guillemin-Urbe [1] keeping in mind that for $n-d+1 \leq j \leq n$, u_j are exchanged by u_j/i .

We have

$$(11) \quad H_1 = \sum_{j=1}^{n-d} u_j(\xi_j^2 + x_j^2) + \sum_{j=n-d+1}^n \frac{1}{i} u_j(\xi_j^2 + x_j^2).$$

As in [1] we introduce complex coordinates, $z_j = x_j + i\xi_j$, with real x_j, ξ_j . In these coordinates $x_j^2 + \xi_j^2 = z_j \bar{z}_j = |z_j|^2$. The Hamiltonian vector field

$$\nu = \sum_j \frac{\partial H_1}{\partial \xi_j} \frac{\partial}{\partial x_j} - \frac{\partial H_1}{\partial x_j} \frac{\partial}{\partial \xi_j}$$

becomes the vector field

$$\frac{2}{i} \sum_{j=1}^{n-d} u_j \left(z_j \frac{\partial}{\partial z_j} - \bar{z}_j \frac{\partial}{\partial \bar{z}_j} \right) - 2 \sum_{j=n-d+1}^n u_j \left(z_j \frac{\partial}{\partial z_j} - \bar{z}_j \frac{\partial}{\partial \bar{z}_j} \right).$$

Then the proof of [1], where u_j for $n-d+1 \leq j \leq n$ are substituted by u_j/i , can be applied and we get inductively that for $N = 1, 2, \dots$ there exists a neighborhood, \mathcal{O} , of $x = \xi = 0$, and a complex canonical transformation, $\kappa : \mathcal{O} \mapsto \mathbb{C}^{2n}$ such that

$$(12) \quad \kappa^* H = \sum_{j=1}^N H_j + R_{N+1} + R'_{N+1},$$

where

- a) The H_j are homogeneous polynomials of degree $2j$ of the form $H_j = h_j(x_1^2 + \xi_1^2, \dots, x_n^2 + \xi_n^2)$, with H_1 given in (11).
- b) R_N is homogeneous of degree $2N$ and of the form $R_N = W_N + R_N^\sharp$, where W_N consists of the terms homogeneous of degree $2N$ in the Taylor series of $V(x_1^2, \dots, x_n^2)$ at $x = 0$, and R_N^\sharp is an artifact of the previous inductive steps.
- c) R'_N vanishes to order $2N + 2$ at the origin and is of the form $R'_N = V - \sum_{k=2}^N V_k + S_N$, where S_N is another artifact of the inductive process. In addition, R'_N is even.

Using this induction argument Guillemin and Uribe show that one can read off from the H_j 's the first N terms in the Taylor expansion of $V(s_1, \dots, s_n)$ at $s = 0$. This argument is invariant under complex scaling. This achieves the proof of Lemma 1. ■

Recalling the tildes introduced by (10) and letting N tend to infinity in (12) we obtain the classical Birkhoff normal form

$$\sum_{j=1}^{\infty} \tilde{H}_j(x_1^2 + \xi_1^2, \dots, x_{n-d}^2 + \xi_{n-d}^2, \tilde{x}_{n-d+1}^2 + \tilde{\xi}_{n-d+1}^2, \dots, \tilde{x}_n^2 + \tilde{\xi}_n^2)$$

with \tilde{H}_1 as in (11). Then after scaling back to $\mathbb{R}^n \times \mathbb{R}^n$ we get the classical Birkhoff normal form as in (9):

$$F_0 = \sum_{j=1}^{\infty} H_j(\xi_1^2 + x_1^2, \dots, \xi_{n-d}^2 + x_{n-d}^2, \xi_{n-d+1}^2 - x_{n-d+1}^2, \dots, \xi_n^2 - x_n^2),$$

with

$$H_1 = \sum_{j=1}^{n-d} u_j(\xi_j^2 + x_j^2) + \sum_{j=n-d+1}^n u_j(\xi_j^2 - x_j^2).$$

The construction of the quantum Birkhoff normal form (7) is well known (see for example [10]).

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INSTITUTE OF MATHEMATICS AND PHYSICS, UNIVERSITY OF WALES, ABERYSTWYTH, PENGLAIS,
CEREDIGION UK SY23 3BZ

E-mail address: `aai@aber.ac.uk`

ON LEAVE FROM MALMÖ HÖGSKOLA, SWEDEN